# §7.2. Blowing up a point in $\mathbb{A}^2$

$$\{\text{smooth projective curves}\}/\underset{\text{iso.}}{\approx} \hookrightarrow \{K \mid \text{f.g. ext. of } k \text{ of tr.deg. } 1\} \overset{1:1}{\longleftrightarrow} \{\text{curves}\}/\underset{\text{birational equiv.}}{\approx}$$

$\Rightarrow$ up to an isomorphism, there is at most one smooth projective curve in each birational equivalent class of curves.

Question: Can we find one, for each class?

Answer. Yes. (need to study blowing up)

For given birational equivalent class, we choose a representation $C$ which is projective plane curve.

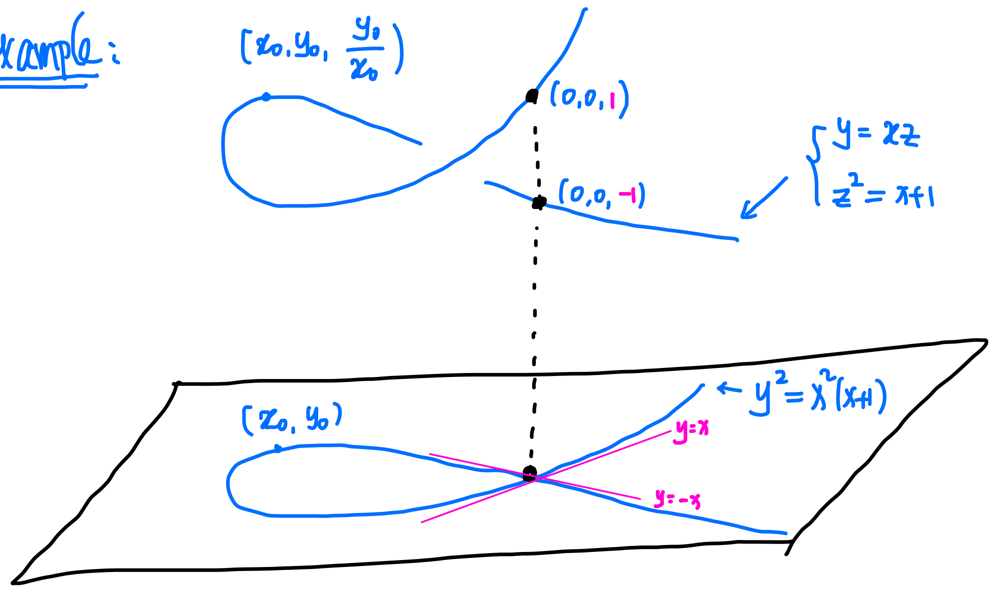
Suppose $\exists$ smooth proj. $C'$ s.t. $C' \sim C$ then $\exists$ morphism

$$C' \to C.$$

to answer the question, we only need to find such a morphism for $C$.

resolve the singularities of a projective curve $C$

**example:**




多值函数

加上额外的信息将重点区分开.（通过方向的不同）

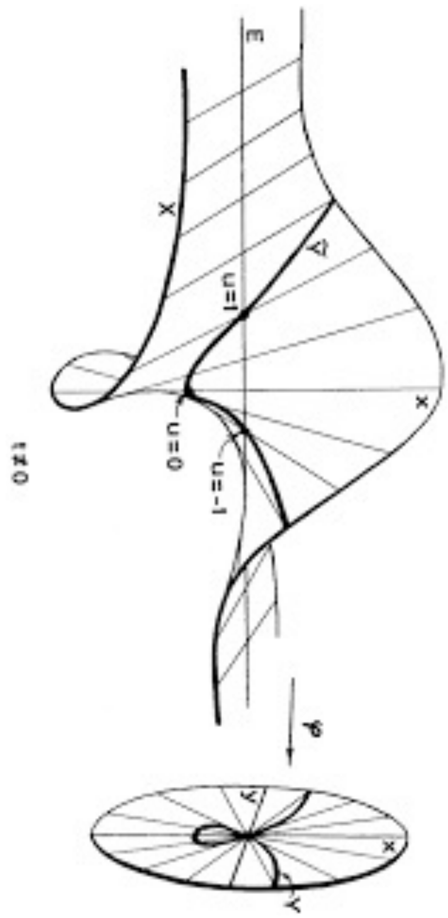
Generally, consider the surface

$$B = V(y - xz) \subseteq \mathbb{A}^3$$

& projection map

$$\pi : B \longrightarrow \mathbb{A}^2$$

For give curve $C \subset \mathbb{A}^2$ with multiple pt at origin. We try to find a curve $C'$ on $B$ s.t. $C'$ is "better" than $C$.



How ~~do~~ under stand $B$ geometrically?

$f: \mathbb{A}^2 \dashrightarrow \mathbb{A}^1 \quad (x,y) \mapsto \frac{y}{x}$ with domain $U = \mathbb{A}^2 \setminus V(x)$.

$\Rightarrow$ Graph of $f$:

$$G = \{(x,y,z) \in \mathbb{A}^3 \mid y = xz,\ x \neq 0\} \subset U \times \mathbb{A}^1 \subset \mathbb{A}^2 \times \mathbb{A}^1 = \mathbb{A}^3.$$

$$\cap$$

$$B := \{(x,y,z) \in \mathbb{A}^3 \mid y = xz\} \hookrightarrow \mathbb{A}^3$$

$\hookrightarrow z = \frac{y}{x}$ ! $\qquad B = \text{var.}$ (since $Y - XZ = \text{irr}$ ! )

$B =$ closure of $G$ in $\mathbb{A}^3$

$\pi : B \longrightarrow \mathbb{A}^2$

$$\begin{tikzcd}
\pi^{-1}(U) = G \arrow[r, hook] \arrow[d, "\cong"] & B \arrow[d, "\pi"] & L = \pi^{-1}(p) \arrow[l, hook'] \arrow[d] \\
U \arrow[r, hook] & \mathbb{A}^2 & \{p\} \arrow[l, hook']
\end{tikzcd}$$

- $\mathrm{im}(\pi) = U \cup \{p\}$
- $\pi^{-1}(p) = \{(0,0,z) \mid z \in k\} =: L$
- $G = B \setminus L$
- $\pi : G \xrightarrow{\cong} U$

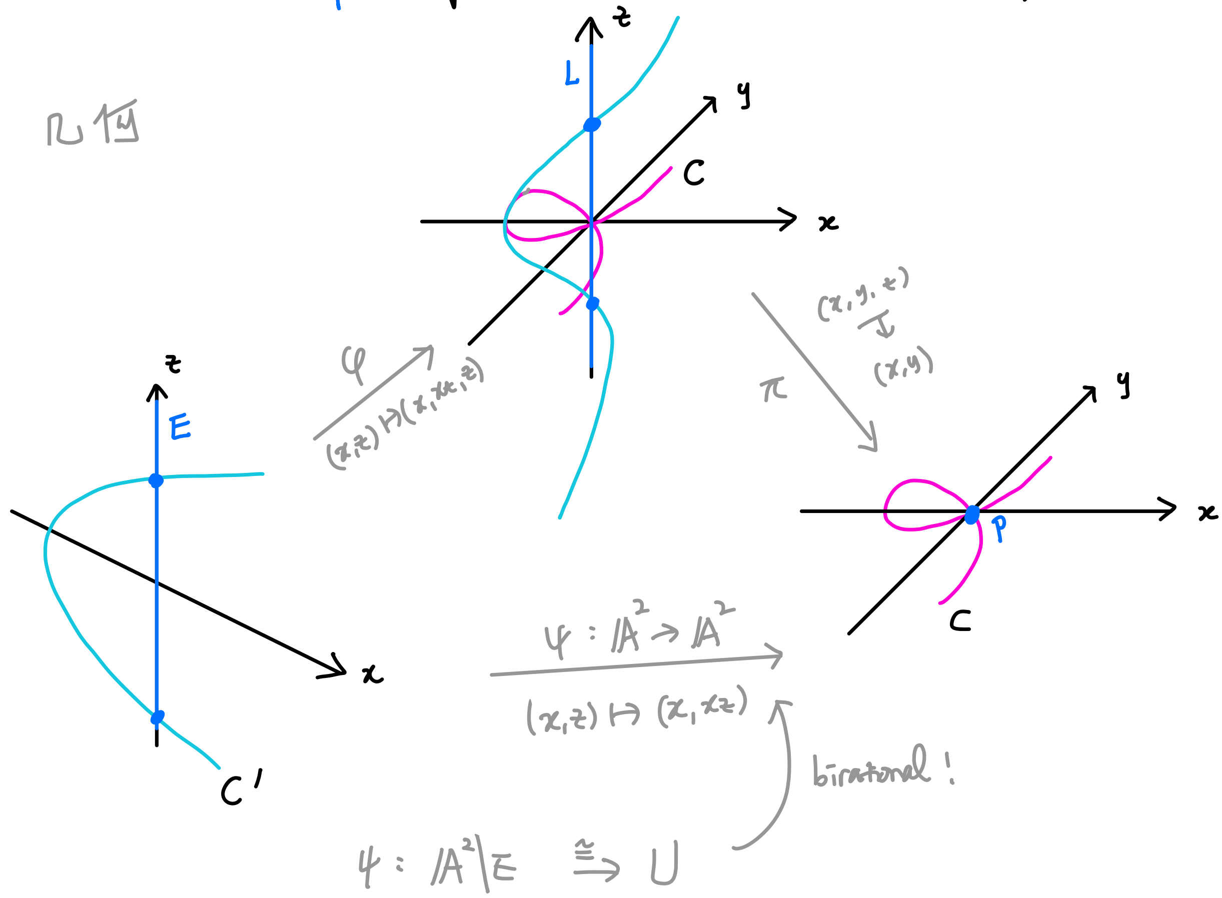
- $\mathbb{A}^2 \underset{\varphi}{\xrightarrow{\cong}} B$

  $(x,z) \longmapsto (x, xz, z)$

通过$\varphi$可将$B$上的曲线拉回平面上.

sketch the real part for curve $C = V(Y^2 - X^2(X+1))$

几何




理论分析 ( $C'$ is better than $C$ ).

$C \subset \mathbb{A}^2$ curve. $\Rightarrow$ $C_0 := C \cap U \hookrightarrow C$.

$\parallel$

$V(X)$ $\Rightarrow$ $C_0' := \psi^{-1}(C_0)$

$\Rightarrow$ $C' :=$ closure of $C_0'$ in $\mathbb{A}^2$

代数

$f : C' \to C$ restriction of $\psi$ to $C'$.

$\Rightarrow$ $f =$ birational morphism of $C'$ to $C$.

i.e. $\tilde{f} : k(C) = k(x,y) \cong k(C') = k(x,z)$

**Lem (1):** $C = V(F)$, $F = F_r + F_{r+1} + \cdots + F_n$ ($F_i$ = form of deg $i$ in $k[X,Y]$)

$r = m_P(C)$. $n = \deg(C)$. Then

$$C' = V(F')$$

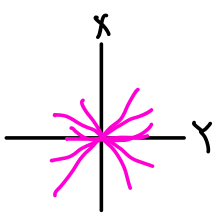
where $F' = F_r(1,Z) + X F_{r+1}(1,Z) + \cdots + X^{n-r} F_n(1,Z)$

Pf:

$$\left.\begin{array}{l} \left.\begin{array}{l} F(X, XZ) = X^r F' \\ F_r(1,Z) \neq 0 \Rightarrow X \nmid F' \end{array}\right\} \Rightarrow C_0' \subset V(F') \\ F = \text{irr} \Rightarrow F' = \text{irr} \quad \Rightarrow V(F') = \text{irr} \end{array}\right\} \Rightarrow V(F') = C'$$

↳ $\left( F' = GH \Rightarrow F = X^r G(X, Y/X) \cdot H(X, Y/X) \right)$

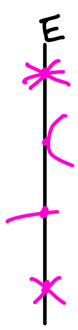
**Assumption:** $X$ is NOT tangent to $C$ at $P$. (coordinate changing)




$\Rightarrow$ (multiplying $F$ by a constant) WMA:

$$F_r = \prod_{i=1}^{s} (\underbrace{Y - \alpha_i X}_{\text{tangents to } F \text{ at } P})^{r_i}$$

**Lem (2)** $f^{-1}(P) = P_1, \cdots, P_s$, where $P_i = (0, \alpha_i)$, and




$$m_{P_i}(C') \leq I(P_i, C' \cap E) = r_i$$

If $P$ is an ordinary mult. pt on $C$, then

$P_i$ simple on $C'$ and $\mathrm{ord}_{P_i}^{C'}(x) = 1$.

Pf: $f^{-1}(P) = C' \cap E = \{ (0, \alpha) \mid F_r(1, \alpha) = 0 \}$

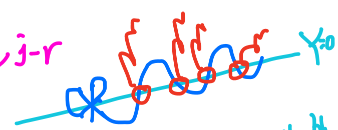
$$m_{P_i}(C') \leq I(P_i, F' \cap X) = I\left(P_i, \prod_{i=1}^{s} (Z - \alpha_i)^{r_i} \cap X\right) = r_i$$

Lem (3). $\exists\ \underset{\substack{\psi \\ p}}{W} \hookrightarrow C$ s.t. $W' = f^{-1}(W) \hookrightarrow C'$ affine open subvar

① . $f(W') = W$

② . $\Gamma(W')/\Gamma(W)$ = finite with $x^{r-1}\Gamma(W') \subset \Gamma(W)$

Pf: $F = \sum_{i+j \geq r} a_{ij} x^i y^j \qquad H = \sum_{j \geq r} a_{0j} y^{j-r}$



$h = H \bmod I(C) \in \Gamma(C)$.

去掉x轴上的除原点以外的与C的交点..

$H(0,0) = 1 \Rightarrow W = C_h \ni p$ open affine in $C$.

$\Rightarrow W' = f^{-1}(W) = C'_h$ open affine in $C'$

To prove ①&②, ONTS: $z$ integral over $\Gamma(W)$. i.e.

$$z^r + b_1 z^{r-1} + \cdots + b_r = 0 \qquad (*)$$

for some $b_1, \dots, b_r \in \Gamma(W)$

$\Bigg($ Since $\Gamma(W') = \Gamma(W)[z] \Rightarrow \Gamma(W') = \sum_{i=0}^{r-1} \Gamma(W) \cdot z^i$

$\left.\begin{matrix} \\ x^{r-1} \cdot z^i \in \Gamma(W) \ \forall i \leq r-1 \end{matrix}\right\} \Rightarrow \checkmark$

for any $(x,y) \in W$ we can solve $*$ ~~to~~ find $(x,z) \in W'$ ! $\Bigg)$

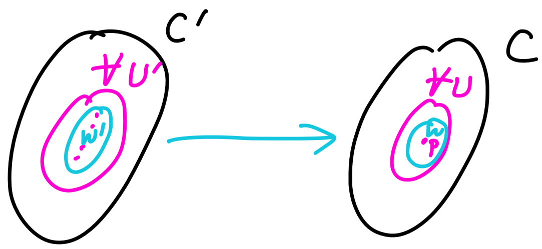
$$F'(x,z) = \sum_{i+j\geq r} a_{ij}\, x^{i+j-r} z^{j} = \sum_{i+j\geq r} a_{ij}\, y^{i+j-r} z^{r-i}$$

$$= \sum_{i=0}^{r-1} \left( a_{ij}\, y^{i+j-r} \right) \cdot z^{r-i} + \sum_{\substack{i\geq r \\ i+j\geq r}} a_{ij}\, x^{i-r} y^{j}$$

$$\begin{cases} b_i = \dfrac{1}{h} \displaystyle\sum_{j} a_{ij}\, y^{i+j-r} & 1 \leq i < r \\ b_r = \dfrac{1}{h} \displaystyle\sum_{\substack{i\geq r \\ j}} a_{ij}\, x^{i-r} y^{j} \end{cases}$$

$$F'(x,z) = 0 \Rightarrow z^r + b_1 z^{r-1} + \dots + b_r = 0 \quad !$$

Rmk: 1) $W, W'$ can be taken arbitrarily small.




2) (linear change of coordinates) assume $W$ includes any f.set of pt on $C$ we wish.

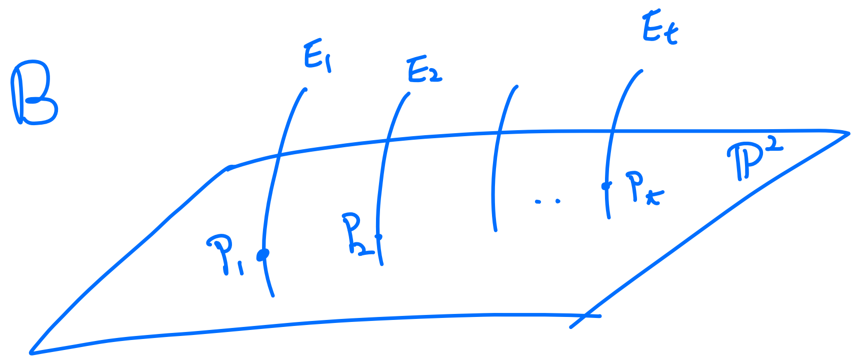
$$(X,Y) \mapsto (X+\alpha Y, Y)$$

$\hookrightarrow$ zeros of $H$: $(X,Y) \mapsto (X, Y+\beta X)$

# §7.3 Blowing up Points in $\mathbb{P}^2$

**aim:** blow up points $P_1, \dots, P_t \in \mathbb{P}^2$. i.e. replace each by a projective line

WMA: $P_i = [a_{i_1} : a_{i_2} : 1] \in U_3 \quad \forall i = 1, \dots, t$.




$U := \mathbb{P}^2 \setminus \{P_1, \dots, P_t\}$. Define

$$f_i : U \to \mathbb{P}^1 \quad [x_1 : x_2 : x_3] \mapsto [x_1 - a_{i_1} x_3 : x_2 - a_{i_2} x_3] \qquad (*)$$

and

$$f = (f_1, f_2, \dots, f_t) : U \to \overbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1}^{t}$$

$$G := \text{graph of } f \subseteq U \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1$$

$X_1, X_2, X_3$ homogeneous coordinates for $\mathbb{P}^2$

$Y_{i1}, Y_{i2}$ homogeneous coordinates for $i$th $\mathbb{P}^1$

$$\mathbb{B} := V\left(\{Y_{i1}(X_2 - a_{i2}X_3) - Y_{i2}(X_1 - a_{i1}X_3) \mid i = 1, \cdots, r\}\right)$$
$$\subseteq \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

**Fact:** $B$ is the closure of $G$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$
(step 6) In particular, $B$ is a variety.

$$\begin{tikzcd} B \arrow[r, hook] \arrow[rr, bend right, "\pi"'] & \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \arrow[r, "Pr"] & \mathbb{P}^2 \end{tikzcd}$$

**Step 1.** $E_i := \{P_i\} \times \{f_1(P_i)\} \times \cdots \times \overset{i\text{-th } \mathbb{P}^1}{\mathbb{P}^1} \times \cdots \times \{f_r(P_i)\} \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$

$$E_i \xrightarrow{\sim} \mathbb{P}^1$$

**Step 2.** $G = B \cap (U \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1) = B \setminus \bigcup_{i=1}^{r} E_i$

In particular,

$$B \setminus \bigcup_{i=1}^{r} E_i \xrightarrow[\cong]{\pi} U$$

**Step 3.** $T$ = projective change of coordinates of $\mathbb{P}^2$.

$P_i' := T(P_i)$

$f_i' : \mathbb{P}^2 \setminus \{P_1', \cdots, P_t'\} \to \mathbb{P}^1$ as in $(*)$.

$f' = (f_1', \cdots, f_t')$, $G', B', E_i'$

Then

1) $\exists!$ proj. change of coordinates $T_i$ of $\mathbb{P}_1$ s.t. $T_i \circ f_i = f_i' \circ T$

2). $(T_1 \times \cdots \times T_t) \circ f = f' \circ T$

3). $G, B, E_i \xrightarrow{T \times T_1 \times \cdots \times T_t} G', B', E_i'$

$$\begin{tikzcd}
U \arrow[r, "T"] \arrow[d, "f = (f_1, \cdots, f_t)"] & U' \arrow[d, "f' = (f_1', \cdots, f_t')"] \\
\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \arrow[r, "T_1 \times \cdots \times T_t"] & \mathbb{P}^1 \times \cdots \times \mathbb{P}^1
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[rrr] \arrow[dr, hook] \arrow[dd] & & & G' \arrow[dl, hook'] \arrow[dd] \\
& U \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \arrow[r, "T \times T_1 \times \cdots \times T_t"] \arrow[dd] & U' \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \arrow[dd] & \\
B \arrow[rrr] \arrow[dr, hook] & & & B' \arrow[dl, hook'] \\
& \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \arrow[r, "T \times T_1 \times \cdots \times T_t"] & \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 &
\end{tikzcd}$$

**Step 4.** $T_i$ = proj. change of coordinates of $\mathbb{P}^1$ (for one $i$). Then $\exists$ proj. change of coordinates $T$ of $\mathbb{P}^2$ s.t.

$$\begin{cases} T(P_i) = P_i \\ f_i \circ T = T_i \circ f_i \end{cases}$$

$$\begin{tikzcd}
U \arrow[r, "T"] \arrow[d, "f_i"] & U \arrow[d, "f_i"] \\
\mathbb{P}^1 \arrow[r, "T_i"] & \mathbb{P}^1
\end{tikzcd}$$

**Step 5.** study the behavior of $\pi$ around some pt $Q \in E_i$.

WLOG: $i=1$ and $P_1 = [0:0:1]$ $\quad Q = [\lambda:1] \in \mathbb{P}^1$ $\lambda \in k$

$$\varphi_3 : \mathbb{A}^2 \xrightarrow{\cong} U_3 \hookrightarrow \mathbb{P}^2 \quad (x,y) \mapsto [x:y:1]$$

$$V := U_3 \setminus \{P_2, \dots, P_t\} \ni P_1$$

$$W := \varphi_3^{-1}(V) \subset \mathbb{A}^2$$

$$\psi : \mathbb{A}^2 \to \mathbb{A}^2 \quad (x,z) \mapsto (x, xz)$$

$$W' := \psi^{-1}(W)$$

$$\varphi : W' \longrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$
$$(x,z) \mapsto \big([x:xz:1], [1:z], f_2([x:xz:1]), \dots, f_t([x:xz:1])\big)$$

Then $\varphi$ is a morphism and

$$\pi \circ \varphi = \varphi_3 \circ \psi.$$

$= B \setminus \left( \bigcup_{i \geq 1} E_i \cup V(X_3) \cup V(Y_{12}) \right)$

$$\begin{tikzcd}
\psi^{-1}(W) =: W' \arrow[rr, "\varphi"] \arrow[dr] \arrow[dd, "\psi"] & & \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \arrow[dd, "\pi"] \\
& V' \hookrightarrow B \arrow[ur, hook] & \\
\varphi_3^{-1}(V) =: W \arrow[rr, "\varphi_3"] & & \mathbb{P}^2
\end{tikzcd}$$

$V = U_3 \setminus \{P_2 \cdots P_t\}$

**Step 6.** $B =$ closure of $G$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ $\quad (\Rightarrow B = \text{var.})$

Pf: $\forall\, S \hookrightarrow \mathbb{P}^2 \times \cdots \times \mathbb{P}^1$ with $G \subseteq S$. then

$\varphi^{-1}(S) \hookrightarrow W'$ & $\varphi^{-1}(S) \supset \varphi^{-1}(G) = W' \setminus V(x) \overset{\text{open}}{\hookrightarrow} W'$

$\Rightarrow \varphi^{-1}(S) = W' \Rightarrow Q \in S \Rightarrow B \subset S.$

Step 7. locally $\pi : B \to \mathbb{P}^2$ looks like $\psi : \mathbb{A}^2 \to \mathbb{A}^2$ (in §7.2)

$$\begin{tikzcd}
\mathbb{A}^2 \supset W' \arrow[r, "\varphi", "\cong"'] \arrow[d, "\psi"'] & V' \subset B \arrow[d, "\pi"] \\
\mathbb{A}^2 \supset W \arrow[r, "\varphi_3", "\cong"'] & V \subset \mathbb{P}^2
\end{tikzcd}$$

The inverse morphism of $\varphi$ is (the restriction to $V'$ of)

$$\mathbb{P}^2 \times \cdots \times \mathbb{P}^1 \setminus V(X_3 Y_{12}) \longrightarrow \mathbb{A}^2$$

$$([x_1 : x_2 : x_3], [y_{11} : y_{12}], \cdots) \longmapsto (x_1/x_3,\ y_{11}/y_{12})$$

Step 8. $C \subset \mathbb{P}^2$ irr curve.

$C_0 := C \cap U$, $\quad C_0' := \pi^{-1}(C_0) \subset G$

$C' :=$ closure of $C_0'$ in $B$.

$\pi \rightsquigarrow f : C' \to C$ (birational morphism)

$$\begin{tikzcd}
C' \arrow[r, "f"] & C \\
C_0' \arrow[u, hook, "\text{open}"] \arrow[r, "\cong"] & C_0 \arrow[u, hook, "\text{open}"']
\end{tikzcd}$$

$f$ looks like the affine map in §7.2.

Prop 1. $C$ = irr. proj. plane curve.
Suppose all multiple pts of $C$ are ordinary. Then
$\exists$ birational morphism $f: C' \to C$
nonsingular projective.

Pf: apply (8) $\Rightarrow$ $f: C' \to C$
step 2 in § 7.2 $\Rightarrow$ $C'$ nonsingular.